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The master integrable two-dimensional system with a quartic second integral

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Abstract

In this paper we construct a new integrable natural conservative mechanical system admitting a second invariant of the fourth degree in velocities. This system is quite general and involves 21 parameters. We also show that all systems with a quartic integral known up to date can be obtained from it as special cases by a relevant choice of the values of parameters. The results are applied to problems of particle and rigid body dynamics. New integrable cases are obtained as special versions of the new system. These cases include motions in a plane, Lobachevsky plane, sphere and surfaces of variable curvature. They also include generalizations of the classical cases of Kovalevskaya, Chaplygin and Goriatchev with the addition of certain singular terms to the potential.

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1. Introduction

It is well known that integrable systems are just a rare exception among Hamiltonian systems. This situation, however, makes them of greater importance. They are the only examples of systems whose behaviour can be studied globally and on an infinite time interval. They can also be used, through perturbation theories, to draw certain conclusions about the motion of nonintegrable systems close to them.

In almost all known integrable problems of mechanics the second integral, which is needed for establishing integrability and solving equations of motion, turns out to be a polynomial in the velocity variables. The classification of all possible systems of this type is a long-standing problem of mechanics. The first trial to construct all plane systems with a linear, quadratic or cubic integral dates back to Bertrand [1], more than 150 years ago. The case of a quadratic integral received a push by Darboux at the turn of the last century [2] (see also [3]).

Kovalevskaya's integrable case of the dynamics of a heavy rigid body moving about a fixed point was probably the first known case of a mechanical system having an integral quartic in velocities in addition to the energy integral [38]. It was followed shortly by the case due to Chaplygin of motion of a body in a liquid [40]. Up to now, a rather limited number of integrable cases of motion of a particle in the Euclidean plane with a quartic integral was found, mostly in the past 20 years (e.g. [22–32]). Most of these cases are listed in Hietarinta's review [10].

None of the previous works was devoted to a systematic search of polynomial integrals in cases where the configuration manifold is not an Euclidean plane. The two cases of Kovalevskaya and Chaplygin have remained until recently the only known examples of natural systems with a quartic integral on a two-dimensional curved manifold.

In virtue of the famous Maupertuis principle, the motion of a natural mechanical system can be brought into equivalence (more precisely, orbital equivalence) with the geodesic flow on some Riemannian metric. Metrics on the Riemannian sphere associated with known integrable cases of rigid body dynamics were constructed in [33]. Two families of integrable systems with a quartic integral on S^2 were obtained in [34, 35]. Few more works discussed possible integrable systems with low-degree polynomials on S^2 and the hyperbolic plane H^2 (see e.g. [11–13]).

The method introduced in our work [6] has proved successful in constructing several new many-parameter families of integrable two-dimensional (not necessarily plane) mechanical systems with integrals quadratic [7] and cubic [9] in velocities. Some of these systems unify and generalize certain previously known ones. In particular, the famous integrable cases of rigid body dynamics are all recovered and mostly generalized by introducing additional parameters into their structure [7, 9].

In the present work we use the same method to construct quite a new general integrable system that seems to include as special cases all known systems to date. The original procedure applied in [6–9] is augmented by a transformation of the independent variable, which allows systematically adding certain extra parameters to the structure of the system, depending on the structure of the potential function of that system. It turned out that this modification of the method enhances the applicability of the results, giving wider possibility of getting curved metrics on the configuration space. In this way we obtain more known integrable systems on manifolds with constant (positive or negative) Gaussian curvatures as well as flat manifolds, as special cases corresponding to particular choices of the set of parameters.

1.1. Primary formulation of the problem

Consider the natural conservative mechanical system described by the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}\dot{q}_i\dot{q}_j - V, \tag{1}$$

where the four symbols a_{ij}, V denote certain functions of the generalized coordinates q_1, q_2 only. This system clearly admits the energy integral

$$I_1 = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}\dot{q}_i\dot{q}_j + V = h, \tag{2}$$

with h the arbitrary energy parameter. It is evident that this system is time reversible. This means that if an integral of motion polynomial in the velocities $\dot{q}_1, \dot{q}_2$ contains even and odd powers of the velocity variables, then the even and odd parts of this integral are both integrals of motion. As we are interested here in systems admitting a quartic integral, the most general form of this integral can be written as

$$I = \sum_{i=0}^{4} C_{4,i}\dot{q}_1^i \dot{q}_2^{4-i} + \sum_{i=0}^{2} C_{2,i}\dot{q}_1^i \dot{q}_2^{2-i} + C_0, \tag{3}$$

where the nine coefficients $C_{4,i}$, $C_{2,i}$ and C_0 are functions in q_1, q_2.

To construct a system of type (1) admitting an integral of the form (3), one should equate to zero the time derivative $\frac{\mathrm{d}I}{\mathrm{d}t}$ in virtue of the equations of motion derived from the Lagrangian (1). This will result in a system of coupled nonlinear partial differential equations involving the 13 unknown functions $\{a_{ij}\}.V$, $\{C_{4,i}\}$, $\{C_{2,i}\}$, C_0. This system is quite complicated. It was solved only under certain restrictive forms of the ansatz (3) and only for $\{a_{ij} = \delta_{ij}\}$, i.e. when the configuration space is an Euclidean plane (see e.g. [10] and also [14–29]). A crucial improvement of this procedure consists in certain steps, which lead to maximal simplification of the forms of the Lagrangian and the integral and to reduce the whole system of governing equations to only four first-order equations involving a minimal number of four unknown coefficients. The last system is eventually reduced to a single nonlinear partial differential equation of the fourth order. Achieving these steps relies on some properties of the system (1) explained in the next subsections.

1.2. The use of isometric coordinates

It is always possible to refer the two-dimensional system to isometric coordinates x, y (say) on the configuration space[1]. We can write the Lagrangian of the system (1) in these coordinates as

$$L = \tfrac{1}{2}\Lambda(\dot{x}^2 + \dot{y}^2) - V, \tag{4}$$

where Λ, V are certain functions of x, y. The energy integral takes the form

$$I_1 = \tfrac{1}{2}\Lambda(\dot{x}^2 + \dot{y}^2) + V = h. \tag{5}$$

In the form (4) the Lagrangian involves only two functions Λ and V instead of four in (1).

1.3. The use of conformal mapping of the plane of the variables

Isometric coordinates on the configuration manifold are not unique. After affecting an arbitrary conformal mapping,

$$x + \mathrm{i}y = z(w), \qquad w = u + \mathrm{i}v \tag{6}$$

the Lagrangian (4) changes to

$$L_0 = \frac{1}{2}\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right|^2 \Lambda[\overset{*}{u}^2 + \overset{*}{v}^2] - V. \tag{7}$$

Thus, the above transformation preserves the structure of (7), changing only Λ to $\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right|^2\Lambda$.

1.4. Change of the independent variable and the energy parameter of a natural system

It is well known (e.g. [4] or [5]) that transforming time t to a new independent variable t_1 by the relation

$$\mathrm{d}t = \Lambda \mathrm{d}t_1, \tag{8}$$

[1] In systems of three dimensions and higher this property is retained only under some conditions on the metric.

the original system (4) is transformed to the one with the Lagrangian

$$L_1 = \tfrac{1}{2}[\overset{*}{x}{}^2 + \overset{*}{y}{}^2] - \Lambda V + h\Lambda \tag{9}$$
$$= \tfrac{1}{2}[\overset{*}{x}{}^2 + \overset{*}{y}{}^2] - V_1, \tag{10}$$

where the asterisk denotes derivative with respect to t_1 and $V_1 = \Lambda(V - h)$.

If the system (9), referred to the new independent variable (the fictitious time), is to be treated in the normal way as a conservative system, it should admit another energy-type integral:

$$\bar{I}_1 = \tfrac{1}{2}[\overset{*}{x}{}^2 + \overset{*}{y}{}^2] + V_1 = \bar{h},$$

involving an arbitrary parameter $\bar{h}$. Returning this integral to the original time variable and dividing by Λ, we get

$$\frac{1}{2}\Lambda(\dot{x}^2 + \dot{y}^2) + V = h + \frac{\bar{h}}{\Lambda}.$$

Comparing this relation to (5), we find that the transformed system (9) is equivalent to the original system only on the zero level of the parameter $\bar{h}$. This means that the free system (4) is consistent to the transformed system (9) under the restriction

$$\bar{I}_1 = \tfrac{1}{2}[\overset{*}{x}{}^2 + \overset{*}{y}{}^2] + V_1 = 0 \tag{11}$$

at the energy of the latter. Different energy levels of the first system are mapped to different values of the parameter h in the transformed system.

This system (9) describes motion of a fictitious particle in the plane under the action of forces with potential V_1 in which the energy constant of the original system already enters linearly as a parameter. This structure of the potential function is characteristic for natural systems of physical and mechanical significance. When the potential is found in an inverse way, as from the solution of a certain system of partial differential equations, it should be expressed in the form (10) involving an arbitrary linear multiplier and then perform the time-variable change inverse to (8) to obtain a system of the type (4) admitting an unconditional integral of the type (5) with arbitrary h.

The general quartic integral (3) is now expressed as the sum of three homogeneous polynomials in $\overset{*}{x}$, $\overset{*}{y}$ involving all 9 coefficients as functions of the coordinates:

$$I = \sum_{i=0}^{4} A_{4,i}\overset{*}{x}{}^i \overset{*}{y}{}^{4-i} + \sum_{i=0}^{2} A_{2,i}\overset{*}{x}{}^i \overset{*}{y}{}^{2-i} + A_0. \tag{12}$$

1.5. Form invariance of the Lagrangian and the equations of motion under conformal mapping of the plane

After affecting an arbitrary conformal mapping

$$x + \mathrm{i}y = z(\zeta), \qquad \zeta = \xi + \mathrm{i}\eta \tag{13}$$

the Lagrangian (9) changes to

$$L_1 = \frac{1}{2}\left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 [\overset{*}{\xi}{}^2 + \overset{*}{\eta}{}^2] - V_1. \tag{14}$$

Applying the independent variable change

$$\mathrm{d}t_1 = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 \mathrm{d}\tau, \tag{15}$$

one can again reduce the Lagrangian to the particle form:

$$L_2 = \frac{1}{2}[\xi'^2 + \eta'^2] + U, \qquad U = -\left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 V_1. \tag{16}$$

where the primes denote derivatives with respect to τ. The corresponding equations of motion are

$$\xi'' = \frac{\partial U}{\partial \xi}, \qquad \eta'' = \frac{\partial U}{\partial \eta}, \tag{17}$$

$$\xi'^2 + \eta'^2 = 2U. \tag{18}$$

It is evident that the above transformation preserves the structure (10) of the potential, changing Λ to $\left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 \Lambda$.

1.6. Simplification of the form of the integral

As has been proved in [6] for the general case of a polynomial integral, using the energy integral (18) and a suitable conformal mapping in the transformation (13), we can always reduce the integral to the form

$$I = \xi'^4 + P\xi'^2 + Q\xi'\eta' + R = \text{const}. \tag{19}$$

This step is crucial for our method, since the integral (19) now involves only three functions P, Q and R of ξ, η instead of nine in the original form (12). Note that conformal mapping followed by a change (15) of time scale leaves invariant the form of equations (17) and (18). It is clear from the above construction that P and Q may depend on the energy parameter h at the most linearly while R is at the most quadratic in h. The whole problem is thus reduced to that of finding the four compatible functions P, Q, R and U.

Let us now consider the system (17) restricted by condition (18). Suppose that this system admits an integral of the form (19). Differentiating (19) with respect to τ and using (17) and (18), we obtain the system of four equations satisfied by the four unknown functions:

$$\frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta} + 4\frac{\partial U}{\partial \xi} = 0, \qquad \frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \xi} = 0, \tag{20}$$

$$\frac{\partial R}{\partial \xi} + 2P\frac{\partial U}{\partial \xi} + Q\frac{\partial U}{\partial \eta} + 2U\frac{\partial Q}{\partial \eta} = 0, \qquad \frac{\partial R}{\partial \eta} + Q\frac{\partial U}{\partial \xi} = 0. \tag{21}$$

From (21) one can express the function R -up to an additive constant- in the form

$$R(\xi, \eta) = -\int Q\frac{\partial U}{\partial \xi}\,\mathrm{d}\eta - \int \left[2P\frac{\partial U}{\partial \xi} + Q\frac{\partial U}{\partial \eta} + 2U\frac{\partial Q}{\partial \eta}\right]_0 \mathrm{d}\xi, \tag{22}$$

where $[]_0$ means that the expression in the bracket is computed for η taking an arbitrary constant value η_0 (say). The whole system (20), (21) is thus reduced to the form of three equations:

$$\begin{gathered}\frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta} + 4\frac{\partial U}{\partial \xi} = 0, \qquad \frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \xi} = 0, \\ \frac{\partial}{\partial \eta}\left(2P\frac{\partial U}{\partial \xi} + Q\frac{\partial U}{\partial \eta} + 2U\frac{\partial Q}{\partial \eta}\right) - \frac{\partial}{\partial \xi}\left(Q\frac{\partial U}{\partial \xi}\right) = 0.\end{gathered} \tag{23}$$

The system (23) can still be reduced to a single nonlinear partial differential equation. In fact, using the substitution

$$P = \frac{\partial^2 F}{\partial \xi^2}, \qquad Q = -\frac{\partial^2 F}{\partial \xi \partial \eta}, \qquad U = -\frac{1}{4}\left(\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2}\right), \tag{24}$$

which satisfies the first two equations identically, we get from the third equation

$$\frac{\partial^2 F}{\partial\xi\partial\eta}\left(\frac{\partial^4 F}{\partial\xi^4}-\frac{\partial^4 F}{\partial\eta^4}\right)+3\left(\frac{\partial^3 F}{\partial\xi^3}\frac{\partial^3 F}{\partial\xi^2\partial\eta}-\frac{\partial^3 F}{\partial\eta^3}\frac{\partial^3 F}{\partial\eta^2\partial\xi}\right)$$
$$+2\left(\frac{\partial^2 F}{\partial\xi^2}\frac{\partial^4 F}{\partial\xi^3\partial\eta}-\frac{\partial^2 F}{\partial\eta^2}\frac{\partial^4 F}{\partial\eta^3\partial\xi}\right)=0. \tag{25}$$

It is not yet known whether equation (25) is integrable, in the sense that some procedure can be pointed out to construct all its solutions. It sounds reasonable to conjecture that this equation is in fact integrable.

The set of solutions of this equation generates all systems of the type (17) having an integral of the form (19) on the zero level of their energy integral. Affecting all possible conformal mappings of the complex $\zeta = \xi + \mathrm{i}\eta$ plane followed by a general point transformation to the generalized coordinates q_1, q_2 with a suitable change of the time variable, we obtain all systems of the general form having a quartic integral on the zero level of their energy integral.

1.7. Classes of integrable systems on an arbitrary energy level

All that we obtain by solving the system (23), or equivalently from a solution of equation (25), is a Lagrangian of the form

$$L = \tfrac{1}{2}(\xi'^2 + \eta'^2) + U, \tag{26}$$

which admits a quartic integral on the zero level of the integral

$$\tfrac{1}{2}(\xi'^2 + \eta'^2) - U = 0. \tag{27}$$

The function U may contain several arbitrary parameters. We shall be interested more in those parameters, which are candidates for being energy parameters, i.e. those which appear only as linear multipliers in a certain term of the modified potential U like h in (9). Each such parameter can be identified as the energy parameter h and its cofactor as the function Λ. The next step is to go back from (9) through the inverse transformation of (8) to the original natural system (4), for which h is the energy parameter.

Thus, suppose that the function U in (26) has the structure

$$U = A_1 v_1 + A_2 v_2 + \cdots - V_0 \tag{28}$$

involving a number of free parameters A_i, $i = 1, 2, \ldots$, which enter only as linear multipliers and do not occur anywhere else in the Lagrangian[2], the functions $V_0(\xi, \eta)$, $v_i(\xi, \eta)$, $i = 1, 2, \ldots$, being also independent of those parameters. In this case each of the parameters $\{A_i\}$ can play the role of an energy parameter with its cofactor as the function Λ in a system of the type (4). It is also possible to construct a single system homotopic to all such systems, in the sense that each of them can be obtained from that system as a special case. The idea is first to introduce new free parameters $h, \alpha_1, a_1, \alpha_2, a_2, \ldots$ by the relations

$$A_1 = \alpha_1 h - a_1, \qquad A_2 = \alpha_2 h - a_2, \ldots. \tag{29}$$

This gives U the form

$$U = (\alpha_1 v_1 + \alpha_2 v_2 + \cdots)h - (a_1 v_1 + a_2 v_2 + \cdots + V_0). \tag{30}$$

The change of the independent variable according to the inverse transformation of (8), i.e.

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\Lambda}, \tag{31}$$

[2] In practice, the metric on the configuration space (equivalently the kinetic energy of the mechanical system) is frequently expressed in terms of some local variables and certain parameters. Such parameters cannot be taken into account even if they also enter in U in the way shown in (28).

where

$$\Lambda = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \tag{32}$$

reduces the Lagrangian (26) to the form

$$L = \frac{1}{2}\Lambda(\dot\xi^2 + \dot\eta^2) + \frac{1}{\Lambda}U \tag{33}$$

$$= \frac{1}{2}(\alpha_1 v_1 + \alpha_2 v_2 + \cdots)(\dot\xi^2 + \dot\eta^2) + h - \frac{a_1 v_1 + a_2 v_2 + \cdots + V_0}{\alpha_1 v_1 + \alpha_2 v_2 + \cdots}. \tag{34}$$

In this Lagrangian, t is the natural time. The additive arbitrary constant h is the energy parameter of the constructed system. The presence of the last parameter in the Lagrangian is now immaterial and it can be discarded. On the other hand, the restriction (27) is replaced by

$$\frac{1}{2}(\alpha_1 v_1 + \alpha_2 v_2 + \cdots)(\dot\xi^2 + \dot\eta^2) + \frac{a_1 v_1 + a_2 v_2 + \cdots + V_0}{\alpha_1 v_1 + \alpha_2 v_2 + \cdots} = h, \tag{35}$$

which demonstrates that the free parameter h is the energy of the system given by (34). In the meantime, the quartic integral (19) after the change (31) to the natural time takes the form

$$I = \Lambda^4 \dot\xi^4 + \Lambda^2 (P\dot\xi^2 + Q\dot\xi\dot\eta) + R = \text{const}. \tag{36}$$

It may happen that the coefficients P, Q and R depend on $\{A_i\}$ and hence on h. This parameter should be replaced wherever it occurs by its expression from (35). Note also that the arbitrary parameters $\{a_i\}$ remain as multipliers in the potential terms, while $\{\alpha_i\}$ enter into the metric on the configuration space and may greatly widen the range of possible interpretation of the system and application of the results. Although splitting the constants $\{A_i\}$ into parts between potential and metric may seem artificial, we will see below some cases of real applications obtained namely in this way.

The idea of introducing extra parameters as in (29) was used to some extent in our previous works [8, 9]. The above construction will be used in section 3 to obtain the main result of this paper.

2. The generic conditional system

In the rest of this paper we shall construct solutions of this system compatible with the assumption that the function F has the structure

$$F = F_1(\xi) + F_2(\eta) + \Psi(\eta)\Phi(\xi). \tag{37}$$

This gives for P, Q and U the expressions

$$\begin{aligned} P &= f(\xi) + \Psi(\eta)\Phi''(\xi), \\ Q &= -\Psi'(\eta)\Phi'(\xi), \\ U &= -\tfrac{1}{4}[f(\xi) + g(\eta) + \Psi(\eta)\Phi''(\xi) + \Psi''(\eta)\Phi(\xi)], \end{aligned} \tag{38}$$

and reduces (25) to the form

$$\begin{aligned} S = {} & [\Psi(\eta)\Phi^{(4)}(\xi) - \Phi(\xi)\Psi^{(4)}(\eta) + f''(\xi) - g''(\eta)]\Psi'(\eta)\Phi'(\xi) \\ & - \{[5\Phi(\xi)\Psi''(\eta) + 2g(\eta)]\Psi'''(\eta) - 3\Psi''(\eta)g'(\eta)\}\Phi'(\xi) \\ & + \{[5\Phi''(\xi)\Psi(\eta) + 2f(\xi)]\Psi'''(\eta) + 3\Phi''(\xi)f'(\xi)\}\Psi'(\eta) \\ = {} & 0. \end{aligned} \tag{39}$$

This equation involving four unknown functions, each depending on one variable, can be reduced to a separable form. The method, based on trial, was successfully used in a previous work to resolve a similar situation [7]. We first note that operating on (39) by the operator

$$\frac{1}{\Phi'(\xi)\Psi'(\eta)}\frac{\partial^2}{\partial\xi\partial\eta}\frac{1}{\Phi'(\xi)\Psi'(\eta)},$$

we get the identity

$$\begin{aligned}&\frac{\Phi^{(5)}(\xi)}{\Phi'(\xi)}+5\frac{\Phi''(\xi)\Phi^{(4)}(\xi)}{\Phi'(\xi)^2}+5\frac{\Phi'''(\xi)^2}{\Phi'(\xi)^2}-5\frac{\Phi''(\xi)^2\Phi'''(\xi)}{\Phi'(\xi)^3}\\&\quad-\left[\frac{\Psi^{(5)}(\eta)}{\Psi'(\eta)}+5\frac{\Psi''(\eta)\Psi^{(4)}(\eta)}{\Psi'(\eta)^2}+5\frac{\Psi'''(\eta)^2}{\Psi'(\eta)^2}-5\frac{\Psi''(\eta)^2\Psi'''(\eta)}{\Psi'(\eta)^3}\right]\\&\quad=0,\end{aligned}\tag{40}$$

which on separation gives

$$\begin{aligned}&\frac{\Phi^{(5)}(\xi)}{\Phi'(\xi)}+5\frac{\Phi''(\xi)\Phi^{(4)}(\xi)}{\Phi'(\xi)^2}+5\frac{\Phi'''(\xi)^2}{\Phi'(\xi)^2}-5\frac{\Phi''(\xi)^2\Phi'''(\xi)}{\Phi'(\xi)^3}=6a_4,\\&\frac{\Psi^{(5)}(\eta)}{\Psi'(\eta)}+5\frac{\Psi''(\eta)\Psi^{(4)}(\eta)}{\Psi'(\eta)^2}+5\frac{\Psi'''(\eta)^2}{\Psi'(\eta)^2}-5\frac{\Psi(\eta)^2\Psi'''(\eta)}{\Psi'(\eta)^3}=6a_4,\end{aligned}\tag{41}$$

where a_4 is an arbitrary constant, i.e. the two functions $\Phi(\xi)$ and $\Psi(\eta)$ satisfy one and the same equation of the fifth order. Multiplying by $\Phi'(\xi)$ and $\Psi'(\eta)$, respectively, in (41) and integrating once, we obtain two fourth-order equations:

$$\begin{aligned}&\Phi^{(4)}(\xi)+5\frac{\Phi''(\xi)\Phi'''(\xi)}{\Phi'(\xi)}-6a_4\Phi(\xi)-\frac{3}{2}a_3=0,\\&\Psi^{(4)}(\eta)+5\frac{\Psi''(\eta)\Psi'''(\eta)}{\Psi'(\eta)}-6a_4\Psi(\eta)-\frac{3}{2}b_3=0.\end{aligned}\tag{42}$$

Substituting $\Phi^{(4)}(\xi)$ and $\Psi^{(4)}(\eta)$ from the last equations into (39) and dividing by $\Phi'(\xi)\Psi'(\eta)$ we arrive at a separable equation, from which we get two equations:

$$\begin{aligned}&f''(\xi)+3\frac{\Phi''(\xi)}{\Phi'(\xi)}f'(\xi)+2\frac{\Phi'''(\xi)}{\Phi'(\xi)}f(\xi)-\frac{3}{2}b_3\Phi(\xi)-8A=0,\\&g''(\eta)+3\frac{\Psi''(\eta)}{\Psi'(\eta)}g'(\eta)+2\frac{\Psi'''(\eta)}{\Psi'(\eta)}g(\eta)-\frac{3}{2}a_3\Psi(\eta)-8A=0,\end{aligned}\tag{43}$$

where A is the arbitrary separation constant. The solution of the last two equations can be readily obtained as

$$\begin{aligned}f(\xi)&=\frac{4C_0+4C_1\Phi(\xi)+4A\Phi(\xi)^2+\frac{1}{4}b_3\Phi(\xi)^3}{\Phi'(\xi)^2},\\g(\eta)&=\frac{4D_0+4D_1\Psi(\eta)+4A\Psi(\eta)^2+\frac{1}{4}a_3\Psi(\eta)^3}{\Psi'(\eta)^2},\end{aligned}\tag{44}$$

where C_1, C_0, D_1, D_0 are arbitrary constants.

On the other hand, integrating (42) twice (after multiplying by $\Phi'(\xi)$ and $\Psi'(\eta)$, respectively, every time) and separating variables we obtain two relations:

$$\begin{aligned}\xi-a_5&=\int^{\Phi(\xi)}\frac{1}{(a_4z^4+a_3z^3+a_2z^2+a_1z+a_0)^{\frac{1}{4}}}\,\mathrm{d}z,\\\eta-b_5&=\int^{\Psi(\eta)}\frac{1}{(a_4z^4+b_3z^3+b_2z^2+b_1z+b_0)^{\frac{1}{4}}}\,\mathrm{d}z.\end{aligned}\tag{45}$$

The functions $\Phi(\xi)$, $\Psi(\eta)$ as obtained by inverting the last relations are in general complicated multi-valued functions. It will be more convenient to use them as generalized coordinates. Introducing the notation $p = \Phi(\xi)$, $q = \Psi(\eta)$ and going back through the above formulae, we express the Lagrangian (26) and the quartic integral I in (19) in terms of the new variables as

$$
\begin{aligned}
L = {}& \frac{1}{2}\left[\frac{p'^2}{\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{q'^2}{\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}\right] \\
& - \left[\frac{\nu b_3p^3 + Ap^2 + C_1p + C_0}{\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{\nu a_3q^3 + Aq^2 + D_1q + D_0}{\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}\right] \\
& - \nu\left[\frac{q(4a_4p^3 + 3a_3p^2 + 2a_2p + a_1)}{\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{p(4a_4q^3 + 3b_3q^2 + 2b_2q + b_1)}{\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}\right]
\end{aligned}
\tag{46}
$$

and

$$
\begin{aligned}
I = {}& \frac{p'^4 + 4[\nu q(4a_4p^3 + 3a_3p^2 + 2a_2p + a_1) + \nu b_3p^3 + Ap^2 + C_1p + C_0]p'^2}{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0} \\
& - 16\nu p'q' - 32\nu^2\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0} \\
& - 16\nu D_1p + 4\frac{K_0(p) + qK_1(p) + q^2K_2(p)}{(a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0)},
\end{aligned}
\tag{47}
$$

where

$$
\begin{aligned}
K_0(p) = {}& -\nu^2\big(4b_2a_4 - b_3^2\big)p^6 + 2\nu(Ab_3 - 2\nu a_3b_2)p^5 + (A^2 + 2\nu b_3C_1 - \nu^2a_2b_2)p^4 \\
& + 2(AC_1 + \nu b_3C_0 - 2\nu^2a_1b_2)p^3 + \big(C_1^2 + 2AC_0 - 4\nu^2a_0b_2\big)p^2 + 2C_0C_1p + C_0^2, \\
K_1(p) = {}& -2\nu[2\nu b_3a_4p^6 + 3\nu a_3b_3p^5 - (2a_4C_1 - Aa_3 - 4\nu a_2b_3)p^4 \\
& + (5\nu a_1b_3 + 2Aa_2 - a_3C_1 - 4a_4C_0)p^3 + 3(2\nu a_0b_3 + Aa_1 - a_3C_0)p^2 \\
& + (4Aa_0 + a_1C_1 - 2a_2C_0)p + 2a_0C_1 - a_1C_0], \\
K_2(p) = {}& -\nu^2\big[8a_4^2p^6 + 12a_3a_4p^5 + 3\big(a_3^2 + 4a_2a_4\big)p^4 + 4(a_2a_3 + 4a_1a_4)p^3 \\
& + 6(a_1a_3 + 4a_0a_4)p^2 + 12a_0a_3p + 4a_0a_2 - a_1^2\big].
\end{aligned}
\tag{48}
$$

Note that the integral (47) is valid only on the zero level of the energy integral

$$
\begin{aligned}
H = {}& \frac{1}{2}\left[\frac{p'^2}{\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{q'^2}{\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}\right] \\
& + \frac{\nu b_3p^3 + Ap^2 + C_1p + C_0}{\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{\nu a_3q^3 + Aq^2 + D_1q + D_0}{\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}} \\
& + \nu\left[\frac{q(4a_4p^3 + 3a_3p^2 + 2a_2p + a_1)}{\sqrt{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{p(4a_4q^3 + 3b_3q^2 + 2b_2q + b_1)}{\sqrt{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}\right] \\
= {}& 0.
\end{aligned}
\tag{49}
$$

The Lagrangian function (26) contains 15 parameters

$$a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, A, C_0, C_1, D_0, D_1, \nu,$$

of which ν is the most decisive. When ν vanishes the system becomes separable and the second integral is quadratic in the existing velocities.

3. The master unconditional integrable system

In the Lagrangian (46) six parameters $A, C_0, C_1, D_0, D_1, \nu$ enter as linear multipliers in the potential part but not anywhere else. Let us, as explained in subsection 1.7, introduce 13 new parameters by renaming these parameters

$$\begin{aligned} C_0 &= h_1 - \alpha_1 h \qquad & C_1 &= h_2 - \alpha_2 h \qquad & A &= h_3 - \alpha_3 h \\ D_0 &= h_4 - \alpha_4 h \qquad & D_1 &= h_5 - \alpha_5 h \qquad & \nu &= h_0 - \alpha_0 h. \end{aligned} \tag{50}$$

Inserting those expressions in (46) and rearranging the terms, we get

$$\begin{aligned} L = \frac{1}{2}&\left[\frac{p'^2}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{q'^2}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right] \\ &- \left[\frac{h_0b_3p^3+h_3p^2+h_2p+h_1}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{h_0a_3q^3+h_3q^2+h_5q+h_4}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right] \\ &- h_0\left[\frac{q(4a_4p^3+3a_3p^2+2a_2p+a_1)}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{p(4a_4q^3+3b_3q^2+2b_2q+b_1)}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right] + \Lambda h, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \Lambda = &\left[\frac{\alpha_0b_3p^3+\alpha_3p^2+\alpha_2p+\alpha_1}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{\alpha_0a_3q^3+\alpha_3q^2+\alpha_5q+\alpha_4}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right] \\ &+ \alpha_0\left[\frac{q(4a_4p^3+3a_3p^2+2a_2p+a_1)}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{p(4a_4q^3+3b_3q^2+2b_2q+b_1)}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right]. \end{aligned} \tag{52}$$

Now, we introduce the change of the time variable according to the rule (31). This transforms the Lagrangian (51) to the form

$$\begin{aligned} L^* = \frac{1}{2}\Lambda&\left[\frac{\dot p^2}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{\dot q^2}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right] \\ &- \frac{1}{\Lambda}\left\{\left[\frac{h_0b_3p^3+h_3p^2+h_2p+h_1}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{h_0a_3q^3+h_3q^2+h_5q+h_4}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right]\right. \\ &\left.+ h_0\left[\frac{q(4a_4p^3+3a_3p^2+2a_2p+a_1)}{\sqrt{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0}} + \frac{p(4a_4q^3+3b_3q^2+2b_2q+b_1)}{\sqrt{a_4q^4+b_3q^3+b_2q^2+b_1q+b_0}}\right]\right\} + h, \end{aligned} \tag{53}$$

which now contains the (discardable) free additive energy parameter h. The quartic integral corresponding to the last Lagrangian can be easily obtained from (47) after affecting the substitution (50) by the change

$$I(p,q,p',q') \to I^* = I(p,q,\Lambda\dot p,\Lambda\dot q). \tag{54}$$

This means that

$$\begin{aligned} I^* = &\frac{\Lambda^4\dot p^4 + 4\Lambda^2[h_0q(4a_4p^3+3a_3p^2+2a_2p+a_1)+h_0b_3p^3+h_3p^2+h_2p+h_1]\dot p^2}{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0} \\ &- 16\Lambda^2(h_0-\alpha_0 h)\dot p\dot q \\ &- h\frac{\alpha_0q(4a_4p^3+3a_3p^2+2a_2p+a_1)+\alpha_0b_3p^3+\alpha_3p^2+\alpha_2p+\alpha_1}{a_4p^4+a_3p^3+a_2p^2+a_1p+a_0} \end{aligned}$$

$$
\begin{aligned}
&- 32(h_0 - \alpha_0 h)^2 \sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0} \\
&- 16(h_0 - \alpha_0 h)(h_5 - \alpha_5 h)p + 4\frac{K_0'(p) + qK_1'(p) + q^2 K_2'(p)}{(a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0)}.
\end{aligned} \tag{55}
$$

$K_0'(p)$, $K_1'(p)$ and $K_2'(p)$ are obtained from $K_0(p)$, $K_1(p)$ and $K_2(p)$ in (48) by the substitution (50).

In its final form (53) the Lagrangian involves 21 parameters apart from the additive constant h which can be ignored in the Lagrangian and adopted as the energy constant. Of these parameters, 15 enter in the kinetic energy (or the line element of the configuration space) and 6 enter only as coefficients in the potential terms. In all the known systems, the one with a quartic integral does not exceed 6 (see e.g. [10] and [14–37]). It should be noted that such a colossal system could be obtained only in virtue of the simplifying steps made in the introduction for both the form of the Lagrangian of the general natural system and the form of its quartic integral.

One may explore the richness of the system (53) just by verifying the fact that it contains, to the best of our knowledge, all the known up-to-date integrable cases with a quartic integral, in both particle dynamics and rigid body dynamics. For example, all systems obtained by using the ansatz

$$
V = u(y) + v(y)\Phi(x) \tag{56}
$$

can be obtained as special cases of our system. In fact, in order that the expression for U in (38) reduces to the ansatz (56), the function $\Phi(x)$ should satisfy the equation $\Phi''(x) + N\Phi(x) = 0$ for some constant N, so that Φ is a trigonometric (or hyperbolic) function. This is exactly the form investigated in [30, 34, 35, 42]. All results in these works are special versions of the present one.

In the next sections, we will give some examples in which this system becomes more tractable and comes out as generalization of some more known systems.

4. Integrable motions on Riemannian two-dimensional manifolds, including the plane

A natural question arises: Under which conditions will the system (53) describe the motion of a particle in the Euclidean plane or on a sphere or, more generally, in a flat 2D space or a space of constant curvature? To answer this question, one must analyse the necessary condition that the Gaussian curvature of the metric corresponding to the kinetic energy

$$
\kappa = -\frac{1}{2G}\left[\sqrt[4]{P_4}\frac{\partial}{\partial p}\left(\sqrt[4]{P_4}\frac{\partial \ln G}{\partial p}\right) + \sqrt[4]{Q_4}\frac{\partial}{\partial q}\left(\sqrt[4]{Q_4}\frac{\partial \ln G}{\partial q}\right)\right], \tag{57}
$$

where $P_4(p)$, $Q_4(q)$ are the two quartic polynomials involved in the above formulae, should vanish identically or take a constant value. It turned out that the condition of zero curvature leads to a large number of equations in the 21 parameters. Every working combination of parameters corresponds to a real integrable system in the plane only if it is possible to choose the remaining parameters such that the kinetic energy becomes a positive definite in the velocities. The last step is to affect a conformal transformation that reduces the metric corresponding to (53) to that of the Euclidean plane. Similar procedures are adapted for cases of constant positive and negative curvature, which can be interpreted as cases of motion on the sphere S^2 and the pseudosphere or the hyperbolic plane H^2. A detailed systematic analysis of these points will be presented in a forthcoming paper.

4.1. Generalizations and variations of some Toda-type systems

Let $a_4 = 1, a_0 = a_1 = a_2 = a_3 = b_0 = b_1 = b_2 = b_3 = 0$. Under the coordinate transformation $p = e^x, q = e^y$, the Lagrangian (53) takes the form

$$L = \frac{1}{2}(2\alpha_3 + \alpha_1 e^{-2x} + \alpha_2 e^{-x} + \alpha_4 e^{-2y} + \alpha_5 e^{-y} + 8\alpha_0 e^{x+y})(\dot{x}^2 + \dot{y}^2) - \frac{2h_3 + h_1 e^{-2x} + h_2 e^{-x} + h_4 e^{-2y} + h_5 e^{-y} + 8h_0 e^{x+y}}{2\alpha_3 + \alpha_1 e^{-2x} + \alpha_2 e^{-x} + \alpha_4 e^{-2y} + \alpha_5 e^{-y} + 8\alpha_0 e^{x+y}}(+h). \quad (58)$$

To help identify or at least isolate certain cases of the configuration manifold of this system, we first note that its Gaussian curvature

$$\kappa = -\frac{1}{2\Lambda}\left[\frac{\partial^2 \ln\Lambda}{\partial x^2} + \frac{\partial^2 \ln\Lambda}{\partial y^2}\right],$$
$$\Lambda = 2\alpha_3 + \alpha_1 e^{-2x} + \alpha_2 e^{-x} + \alpha_4 e^{-2y} + \alpha_5 e^{-y} + 8\alpha_0 e^{x+y}. \quad (59)$$

That is,

$$\kappa = -\frac{1}{2\left(2\alpha_3 + \alpha_1 e^{-2x} + \alpha_2 e^{-x} + \alpha_4 e^{-2y} + \alpha_5 e^{-y} + 8\alpha_0 e^{x+y}\right)^3} \times (8\alpha_1 e^{-2x}\alpha_3 + 2\alpha_2 e^{-x}\alpha_3 + 32\alpha_0 e^{x+y}\alpha_3 + 8\alpha_4 e^{-2y}\alpha_3 + 2\alpha_5 e^{-y}\alpha_3 + \alpha_1\alpha_2 e^{-3x} + 8\alpha_1\alpha_4 e^{-2x-2y} + 5\alpha_1\alpha_5 e^{-2x-y} + 80\alpha_1\alpha_0 e^{-x+y} + 5\alpha_2\alpha_4 e^{-x-2y} + 2\alpha_2\alpha_5 e^{-x-y} + 40\alpha_2\alpha_0 e^{y} + 80\alpha_0\alpha_4 e^{x-y} + 40\alpha_0\alpha_5 e^{x} + \alpha_4\alpha_5 e^{-3y}). \quad (60)$$

This curvature vanishes when all but one of the parameters $\alpha_i, i = 0, \ldots, 5$ vanish. We thus get six candidates of integrable Lagrangians in the Euclidean plane. Due to apparent symmetry between the two variables, we have only four distinct cases which we present briefly as follows.

(1) $\alpha_3 = \frac{1}{2}$ and all other α's vanish. We arrive at the system whose Lagrangian is

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - (A_1 e^{-x} + A_2 e^{-2x} + B_1 e^{-y} + B_2 e^{-2y} + C e^{x+y}). \quad (61)$$

It admits the quartic integral

$$I = [\dot{y}^2 + 2(B_1 e^{-y} + B_2 e^{-2y})][\dot{x}^2 + 2(A_1 e^{-x} + A_2 e^{-2x})] + 2C e^{x+y}\dot{x}\dot{y} + C^2 e^{2(x+y)} + 2C(A_1 e^{y} + B_1 e^{x}). \quad (62)$$

It is not hard to see that this system generalizes and unifies all the Toda-type systems found in [14] and [15] (see also [10]), where at most three of the five coefficients of the potential were present but had specific numerical values.

(2) $\alpha_4 = 1$ and all other α's vanish. Substituting $y = -\ln(r), x = \theta$ we get the Lagrangian

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \left(\frac{h_5}{r} + \frac{2h_3 + h_2 e^{-\theta} + h_1 e^{-2\theta}}{r^2} + \frac{8h_0 e^{\theta}}{r^3}\right). \quad (63)$$

The potential in this case can be regarded as a generalization of the centrally symmetric Manev potential $\frac{h_5}{r} + \frac{2h_3}{r^2}$.

(3) When $\alpha_0 = 1$ and all other α's vanish the substitution $x = \ln r + \theta, y = \ln r - \theta$ transforms (58) to

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \left(\frac{A_1}{r^2} + \frac{A_2 e^{\theta} + A_3 e^{-\theta}}{r^3} + \frac{A_4 e^{2\theta} + A_5 e^{-2\theta}}{r^4}\right). \quad (64)$$

This case coincides with that given in table *V* of [37].

(4) When $\alpha_5 = 1$ and all other α's vanish the substitution $x = 2\theta$, $y = -2\ln r$ transforms (58) to

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \left(A_0 r^2 + \frac{A_1 + A_2\,\mathrm{e}^{-2\theta} + A_3\,\mathrm{e}^{-4\theta}}{r^2} + \frac{A_4\,\mathrm{e}^{2\theta}}{r^4}\right). \tag{65}$$

4.2. Generalization of the cases of Bozis and Wojciechowski

Let $a_4 = a_0 = b_0 = 1, a_2 = b_2 = -2, a_1 = a_3 = b_1 = b_3 = 0$. Under the coordinate transformation $p = \sin y, q = \sin x$, the Lagrangian (53) takes the form

$$\begin{aligned} L = {} & \frac{1}{2}\left[-2\alpha_3 - 8\alpha_0 \sin x \sin y + \frac{\beta_1 + \alpha_5 \sin x}{\cos^2 x} + \frac{\beta_2 + \alpha_2 \sin y}{\cos^2 y}\right](\dot{x}^2 + \dot{y}^2) \\ & - \frac{-2h_3 - 8h_0 \sin x \sin y + \frac{h_3 + h_4 + h_5 \sin x}{\cos^2 x} + \frac{h_3 + h_1 + h_2 \sin y}{\cos^2 y}}{-2\alpha_3 - 8\alpha_0 \sin x \sin y + \frac{\beta_1 + \alpha_5 \sin x}{\cos^2 x} + \frac{\beta_2 + \alpha_2 \sin y}{\cos^2 y}}. \end{aligned} \tag{66}$$

When $\alpha_3 = -\frac{1}{2}, \alpha_0 = \alpha_2 = \alpha_5 = \beta_1 = \beta_2 = 0$, we have, after ignoring an insignificant additive constant,

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left(-8h_0 \sin x \sin y + \frac{h_4 + h_5 \sin x}{\cos^2 x} + \frac{h_1 + h_2 \sin y}{\cos^2 y}\right). \tag{67}$$

This gives the system found by Bozis [24]. A slight variation of the parameters in (66) to be $a_2 = b_2 = 2$ changes trigonometric functions to hyperbolic (or exponential) functions, and thus giving solutions of the type of [27].

In a similar way, one can obtain a mix of the two types. Let $a_4 = 1, a_0 = a_1 = a_2 = a_3 = b_3 = b_1 = 0, b_0 = 1, b_2 = -2$. Under the coordinate transformation $p = \sin x, q = e^y$, the Lagrangian (53) takes the form

$$L = \frac{1}{2}\left(\alpha_1\,\mathrm{e}^{-2y} + \alpha_2\,\mathrm{e}^{-y} + \frac{\alpha_3 + \alpha_5 \sin x}{\cos^2 x}\right)(\dot{x}^2 + \dot{y}^2) - \frac{h_1\,\mathrm{e}^{-2y} + h_2\,\mathrm{e}^{-y} + \frac{h_3 + h_5 \sin x}{\cos^2 x}}{\alpha_1\,\mathrm{e}^{-2y} + \alpha_2\,\mathrm{e}^{-y} + \frac{\alpha_3 + \alpha_5 \sin x}{\cos^2 x}}. \tag{68}$$

4.3. A motion in the plane and generalization

Let $a_3 = b_3 = 1, a_0 = a_1 = a_2 = a_4 = b_0 = b_1 = b_2 = 0$. The integrable Lagrangian takes the form

$$L = \frac{1}{2}\left[\alpha_0 r^6 + \alpha_3 r^2 + \frac{\alpha_1}{x^6} + \frac{\alpha_2}{x^2} + \frac{\alpha_4}{y^6} + \frac{\alpha_5}{y^2}\right](\dot{x}^2 + \dot{y}^2) - \frac{k_0 r^6 + k_3 r^2 + \frac{k_1}{x^6} + \frac{k_2}{x^2} + \frac{k_4}{y^6} + \frac{k_5}{y^2}}{\alpha_0 r^6 + \alpha_3 r^2 + \frac{\alpha_1}{x^6} + \frac{\alpha_2}{x^2} + \frac{\alpha_4}{y^6} + \frac{\alpha_5}{y^2}}, \tag{69}$$

where $r = \sqrt{x^2 + y^2}$. This form can be transformed using the mapping

$$x + \mathrm{i}y \to \sqrt{x + \mathrm{i}y} \qquad \text{or} \qquad (x, y) \to \left(\sqrt{\frac{r + x}{2}}, \sqrt{\frac{r - x}{2}}\right) \tag{70}$$

to

$$\begin{aligned} L = {} & \frac{1}{2}\left\{A_0 r^2 + A_1 + \frac{A_2}{y^2} + \frac{A_3(4x^2 + y^2)}{y^6} + \frac{x}{r}\left[\frac{A_4}{y^2} + \frac{A_5(4x^2 + 3y^2)}{y^6}\right]\right\}(\dot{x}^2 + \dot{y}^2) \\ & - \frac{n_0 r^2 + n_1 + \frac{n_2}{y^2} + \frac{n_3(4x^2 + y^2)}{y^6} + \frac{x}{r}\left[\frac{n_4}{y^2} + \frac{n_5(4x^2 + 3y^2)}{y^6}\right]}{A_0 r^2 + A_1 + \frac{A_2}{y^2} + \frac{A_3(4x^2 + y^2)}{y^6} + \frac{x}{r}\left[\frac{A_4}{y^2} + \frac{A_5(4x^2 + 3y^2)}{y^6}\right]}. \end{aligned} \tag{71}$$

The last system contains as a special case $A_0 = A_2 = A_3 = A_4 = A_5 = 0$; $A_1 = 1$ is a case of motion in the Euclidean plane with the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left\{ n_0 r^2 + n_1 + \frac{n_2}{y^2} + \frac{n_3(4x^2+y^2)}{y^6} + \frac{x}{r}\left[\frac{n_4}{y^2} + \frac{n_5(4x^2+3y^2)}{y^6}\right]\right\}. \tag{72}$$

The transformation (70) applied to (69) can be repeated in (71) for $n_4 = n_5 = 0$ to yield the new system

$$L = \frac{1}{2}\left[A_0 + \frac{A_1}{r} + \frac{2A_2(r+x)}{y^2 r} + \frac{4A_3(5r+3x)(r+x)^3}{y^6 r}\right](\dot{x}^2 + \dot{y}^2)$$
$$- \frac{n_0 + \frac{n_1}{r} + \frac{2n_2(r+x)}{y^2 r} + \frac{4n_3(5r+3x)(r+x)^3}{y^6 r}}{A_0 + \frac{A_1}{r} + \frac{2A_2(r+x)}{y^2 r} + \frac{4A_3(5r+3x)(r+x)^3}{y^6 r}}. \tag{73}$$

When $A_1 = A_2 = A_3 = 0$ this system reduces to the plane motion of a particle in the potential

$$V = \frac{n_1}{r} + \frac{2n_2(r+x)}{y^2 r} + \frac{4n_3(5r+3x)(r+x)^3}{y^6 r}. \tag{74}$$

4.4. The Wojciechowska–Wojciechowski system

Let $a_1 = a_2 = a_3 = a_4 = \alpha_0 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, $a_0 = 1$, $\alpha_1 = 1$. After renaming $\xi \to y$, $\eta = x$, the Lagrangian of the system may be written in the form

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Bigg[h_0 b_3 y^3 + h_3 y^2 + h_2 y$$
$$+ \frac{h_0 y(3b_3 q^2 + 2b_2 q + b_1)}{\sqrt{b_3 q^3 + b_2 q^2 + b_1 q + b_0}} + \frac{h_3 q^2 + h_5 q + h_4}{\sqrt{b_3 q^3 + b_2 q^2 + b_1 q + b_0}}\Bigg], \tag{75}$$

where q can be expressed by any real branch of the inverse function of the integral

$$x = \int \frac{\mathrm{d}q}{\sqrt[4]{b_3 q^3 + b_2 q^2 + b_1 q + b_0}}. \tag{76}$$

Without loss of generality one can assume $b_3 > 0$. The function under the quartic root has one real root q_0 or three real roots q_0, q_1, q_2 ($q_0 > q_1 > q_2$). In the first case, the function q has one real branch passing through (x_0, q_0). This branch is clearly unbounded, even in $(x - x_0)$ and changes from ∞ to q_0 and then to ∞ as x changes on $(-\infty, \infty)$. In the second case, in addition to the above branch we have a bounded branch with range $[q_2, q_1]$ which is periodic with the period

$$X = 2\int_{q_2}^{q_1} \frac{\mathrm{d}q}{\sqrt[4]{b_3 q^3 + b_2 q^2 + b_1 q + b_0}}. \tag{77}$$

In [26] Wojciechowska and Wojciechowski pointed out a family of potentials allowing a quartic integral in the problem of plane motion of a particle. These potentials have the general form [26]

$$V = k\left[\gamma(x) - n'(x)y + \tfrac{1}{6}by^3\tfrac{1}{2}ay^2 + ey\right], \tag{78}$$

where n is the solution of

$$n(n''' + b) + 5n'n'' = 0, \tag{79}$$

and γ satisfies the linear equation

$$n(\gamma'' - a) + 3n'\gamma' + 2n''\gamma = 0. \tag{80}$$

The trials in [26] to obtain an explicit solution of (79) gave fragmental and very complicated results. We will now show that the difficulty was caused by an unlucky choice of n as the dependent variable. In fact, if in (79) we replace n by $\Phi'(x)$, rename b as $-3a_3$ and note that $a_4 = 0$, we just obtain equation (42), for which the general solution is given by (45). The problem that forced the authors to enter a maze of special cases in solving (79) can now be explained: expressing the derivative $\Phi'(x)$,

$$\Phi'(x) = \sqrt[4]{b_3\Phi(x)^3 + b_2\Phi(x)^2 + b_1\Phi(x) + b_0},$$

explicitly in terms of x, is a much more complicated task than solving this equation in $\Phi(x)$. Equation (80) can also be identified with the appropriate one in (43). Thus, we conclude that the system described by (75) is just a realization of the system pointed out in [26] by completing the missing steps in its solution.

The special case of (75), when $b_0 = b_1 = b_2 = 0$, $b_3 = 1$, reproduces the potential

$$V = a(x^2 + 16y^2) + b(3x^2y + 16y^3) + cy + \frac{\mathrm{d}}{x^2} + \frac{e}{x^6} \tag{81}$$

obtained in [26] and generalizes earlier results of [16, 17, 20] concerning extensions of integrable versions of the Henon–Heiles system [21].

4.5. Generalization of a system due to Ramani and Grammaticos

Let $a_0 = a_1 = a_3 = a_4 = \alpha_0 = \alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, $a_2 = 1$, $\alpha_2 = 1$. The Lagrangian can be written in the form

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left(\frac{1}{16}h_0b_3y^4 + \frac{1}{4}h_3y^2 + \frac{4h_1}{y^2} + 2h_0q\right) - \frac{h_0y^2(3b_3q^2 + 2b_2q + b_1)}{4\sqrt{b_3q^3 + b_2q^2 + b_1q + b_0}} - \frac{h_3q^2 + h_5q + h_4}{\sqrt{b_3q^3 + b_2q^2 + b_1q + b_0}}(+h), \tag{82}$$

where q is given by the same relation as (76). This system is similar to the previous one in structure. It differs only in the powers of y in some terms.

The system (82) includes some previously known cases as special cases. If we put $b_0 = b_1 = b_2 = 0$, $b_3 = 1$, we obtain, after a simple change of variables and parameters, the plane potential

$$V = (x^2 + 4y^2)[a + b(x^2 + 2y^2)] + \frac{c}{y^2} + \frac{d}{x^2} + \frac{e}{x^6} \tag{83}$$

found in [16]. If, moreover, $a = c = d = e = 0$ we get the potential found in [18] and [19].

5. Two new integrable cases in rigid body dynamics

One of the unexpected results is the applicability of the new system to the field of dynamics of a rigid body about a fixed point under the action of an axisymmetric potential field. After ignoring the cyclic angle of precession on the zero level of the cyclic integral and in the absence of gyroscopic forces, the Routhian of this mechanical system expressed in the other two Eulerian angles has the form (see e.g. [9])

$$R = \frac{1}{2}\left[\dot{\theta}^2 + \frac{C\sin^2\theta\dot{\varphi}^2}{A - (A - C)\cos^2\theta}\right] - \frac{1}{A}V, \tag{84}$$

where θ is the angle of nutation and φ is the angle of proper rotation (about the axis of symmetry of the body). Comparing the structure of this Routhian function to that of the Lagrangian (53) and recalling the procedure followed in a similar situation in [42], we get convinced that they become identical only in the four cases of that work valid for $A = B = 2C$. We complete the analysis here only in the two cases in which the potential is 2π-periodic in the angle of proper rotation φ and is thus single valued on the configuration space of the rigid body.

5.1. *The first case*

Setting

$$\alpha_0 = \alpha_4 = \alpha_5 = 0, \qquad \alpha_1 = -\tfrac{1}{8}, \qquad \alpha_2 = 1, \qquad \alpha_3 = -2$$
$$a_0 = -\tfrac{3}{16}, \qquad a_1 = 2, \qquad a_2 = -6, \qquad a_3 = b_3 = b_1 = 0,$$
$$a_4 = 16, \qquad b_0 = 16a^4, \qquad b_2 = -32a^2,$$

and renaming other constants and performing the substitution

$$p = -\tfrac{1}{4}(\cos^2\theta - \csc^2\theta), \qquad q = a\sin 2\varphi,$$

we obtain the Routhian (84) with the new integrable potential

$$V = A\left[\frac{1}{2}a\sin^2\theta\sin 2\varphi + \frac{\lambda}{\cos^2\theta} - \frac{A_1\sin^2\theta}{\cos^6\theta} + \frac{1+\sin^2\theta}{\sin^2\theta\cos^2 2\varphi}(e_0 + e_1 a\sin 2\varphi)\right], \tag{85}$$

admitting the quartic integral

$$\begin{aligned} I = {} & \frac{\sin^8\theta\dot\varphi^4}{(1+\sin^2\theta)^4} + \left[-2h\sin^2\theta + a\sin 2\varphi(\sin^2\theta + \cos^4\theta) + 4\sin^2\theta\frac{e_0 + e_1\sin 2\varphi}{\cos^2 2\varphi}\right] \\ & \times \frac{\sin^2\theta\dot\varphi^2}{(1+\sin^2\theta)^2} - a\frac{\sin\theta\cos^3\theta\cos 2\varphi\dot\theta\dot\varphi}{(1+\sin^2\theta)} + a(\lambda - h)\sin 2\varphi - \frac{1}{4}a^2(1-\cos^4\theta)\cos^2 2\varphi \\ & + a(\cos^2\theta - \csc^2\theta)(e_1 - 2\frac{(e_1 + e_0\sin 2\varphi)}{\cos^2 2\varphi}) - 4\frac{h(e_0 + e_1\sin 2\varphi) + e_1^2}{\cos^2 2\varphi} \\ & + \frac{4\left(e_0^2 + e_1^2 + 2e_0e_1\sin 2\varphi\right)}{\cos^4 2\varphi}. \end{aligned} \tag{86}$$

This new case (85) involves two arbitrary parameters e_0, e_1 more than the case pointed out recently in [41]. When $e_0 = e_1 = A_1 = 0$, we get a potential due to Goriatchev [39, 36]. If, moreover, $\lambda = 0$ the present case reduces to the well-known case of motion of a body by inertia in a liquid due to Chaplygin [40].

5.2. *The second case*

In this case, we set

$$\begin{aligned} & \alpha_0 = \alpha_1 = \alpha_2 = \alpha_5 = 0, \qquad \alpha_3 = -\frac{1}{2}, \qquad \alpha_4 = \frac{a^2}{2} \\ & h_0 = -\frac{1}{4}, \qquad h_1 = \frac{\lambda}{4}, \qquad h_2 = -\frac{\varepsilon}{2}, \qquad h_3 = 0, \qquad h_4 = a^2\nu_0, \qquad h_5 = a\nu_1 \\ & a_0 = a_1 = a_3 = b_1 = b_3 = 0, \qquad a_2 = a_4 = 1, \qquad b_0 = a^4, \qquad b_2 = -2a^2 \\ & p = -\frac{\cos^2\theta}{2\sin\theta}, \qquad q = a\sin\varphi. \end{aligned} \tag{87}$$

This leads to the new integrable potential

$$V = a\sin\theta\sin\varphi + \frac{\lambda}{\cos^2\theta} + \frac{\varepsilon}{\sin\theta} + \frac{1+\sin^2\theta}{\sin^2\theta\cos^2\varphi}(\nu_0 + \nu_1\sin\varphi). \tag{88}$$

The integral can be written analogously in the form

$$
\begin{aligned}
I = {} & \frac{\sin^8\theta\dot{\varphi}^4}{(1+\sin^2\theta)^4} + 2a\frac{\sin^2\theta\cos\theta\cos\varphi\dot{\theta}\dot{\varphi}}{(1+\sin^2\theta)} \\
& - \left[2h\sin^2\theta + 2a\sin\theta\cos^2\theta\sin\varphi + 4\sin^2\theta\frac{\nu_0+\nu_1\sin\varphi}{\cos^2\varphi}\right]\frac{\sin^2\theta\dot{\varphi}^2}{(1+\sin^2\theta)^2} \\
& + a^2\sin^2\theta\cos^2\varphi - 2a\varepsilon\sin\varphi + 2a\nu_1\frac{\cos^2\theta}{\sin\theta} - 4\frac{\nu_1^2}{\cos^2\varphi} - 4h\frac{\nu_0+\nu_1\sin\varphi}{\cos^2\varphi} \\
& - 4a\frac{\cos^2\theta(\nu_0\sin\varphi+\nu_1)}{\sin\theta\cos^2\varphi} + \frac{4(\nu_0^2+\nu_1^2+2\nu_0\nu_1\sin\varphi)}{\cos^4\varphi}.
\end{aligned}
\tag{89}
$$

This case is also new. It involves the two parameters ν_0, ν_1 more than the case of [30] (see also [36]).

As was done with most of the key formulae in this paper, the constancy of the integrals (86), (89) on the energy level h and zero level of the cyclic constant in virtue of the corresponding Lagrangian equations of motion in each case has been checked directly using computer algebra programs.

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